

DESIGN SENSITIVITY ANALYSIS OF NONLINEAR

STRUCTURAL RESPONSE

J.B. Cardoso and J.S. Arora
Optimal Design Laboratory
College of Engineering
The University of Iowa
Iowa City, IA

ABSTRACT

This paper describes a unified theory of design sensitivity analysis of linear and nonlinear structures for shape, nonshape and material selection problems. The concepts of reference volume and adjoint structure are used to develop the unified viewpoint. A general formula for design sensitivity analysis is derived. Simple analytical linear and nonlinear examples are used to interpret various terms of the formula and demonstrate its use.

1. INTRODUCTION

Design sensitivity analysis gives trend information that can be used in the conventional or optimal design process. The subject, therefore, has received considerable attention in recent years. For a thorough review of the subject Refs. 1 and 2 should be consulted.

The present paper describes a unified variational theory of design sensitivity analysis of linear and nonlinear structures (geometric as well as physical nonlinearities) including shape, nonshape and material selection problems. The adjoint variable approach is utilized although the direct differentiation method can be also easily developed. In Section 2, equations of continuum mechanics for nonlinear analysis are summarized. They are needed in design sensitivity analysis. A unified viewpoint for shape and nonshape design sensitivity analysis is described in Section 3. The concept of a reference volume is explained in Section 4. The variational theory of design sensitivity analysis using adjoint variable approach is developed in Section 5. The theory is used to solve several simple analytical problems in Section 6. Finally concluding remarks are given in Section 7.

2. NONLINEAR ANALYSIS

Nonlinearities in structural systems can be due to large displacements, large strains, material behavior and boundary conditions. Consistent theories to treat these nonlinearities have been developed^{3,4}. We will use the developments and notations of Ref. 4, and follow the Total Lagrangian (or Lagrangian) formulation, although updated Lagrangian formulation can also be used. One of the major difficulties in describing nonlinear analysis is the complexity of notation. We will mostly use standard symbols from the literature for various quantities. Matrix and

tensor notations will be used. One major departure from linear analysis is that quantities must be measured in a deformed configuration. Also, a reference configuration for the quantities must be defined. We will use a left superscript to indicate the configuration in which the quantity occurs and a left subscript to indicate the reference configuration.

A starting point for theory of nonlinear analysis is the principle of virtual work for the body in the deformed configuration at time t (load level t):

$$\int_{0V} {}^t_0 \mathbf{S} \cdot \delta {}^t_0 \boldsymbol{\epsilon} {}^0 dV - \int_{0V} {}^t_0 \mathbf{f} \cdot \delta {}^t_0 \mathbf{u} {}^0 dV - \int_{0\Gamma_T} {}^t_0 \mathbf{T} \cdot \delta {}^t_0 \mathbf{u} {}^0 d\Gamma_T = 0 \quad (1)$$

where left subscript 0 refers to the undeformed configuration, a '.' refers to the standard tensor product and

0V = undeformed volume of the body

${}^t_0 \mathbf{S}$ = Second Piola-Kirchhoff stress tensor

${}^t_0 \boldsymbol{\epsilon}$ = Green-Lagrange strain tensor

${}^t_0 \mathbf{f}$ = body force per unit volume

${}^t_0 \mathbf{u}$ = displacement field

${}^t_0 \mathbf{T}$ = surface traction specified on part of the surface Γ_T

${}^0\Gamma$ = surface of the body

δ = variation in the state fields

Let \mathbf{u}^0 be the specified displacement on the part Γ_u of the surface. The variations of the state fields in Eq. (1) are arbitrary but kinematically admissible. They can be replaced by any kinematically admissible fields. In particular they will be replaced by adjoint structure state fields in later derivations. The virtual work equation can also be written using Cauchy stress tensor and other quantities referred to the deformed configuration. Transformation can be used to recover Cauchy stresses from second Piola-Kirchhoff stresses and vice versa. However, in all the derivations given in this paper we will use the undeformed configuration as the reference configuration.

The Green-Lagrange strain tensor is given as

$${}^t_0 \boldsymbol{\epsilon} = \frac{1}{2} [{}^t_0 \nabla \mathbf{u}^T + ({}^t_0 \nabla \mathbf{u}^T)^T + ({}^t_0 \nabla \mathbf{u}^T) ({}^t_0 \nabla \mathbf{u}^T)^T] \quad (2)$$

The nonlinear stress strain law, in general, can be written as

$${}^t_0 \mathbf{S} = \boldsymbol{\phi}({}^t_0 \boldsymbol{\epsilon}, b) \quad (3)$$

where b is a design variable. Note that for many applications, functional form for $\boldsymbol{\phi}$ is not known. In numerical implementations, the explicit form is not needed. Only an incremental stress-strain relation is required.

Equations (1) to (3) are nonlinear in the displacement field ${}^t\mathbf{u}$. There are several methods for solving such system of equations.⁵ The incremental/iterative procedure based on Newton methods is the most commonly used and effective procedure. This will be summarized here. In the derivation of the procedure, it is assumed that equilibrium is known at t and it is desired at $t+\Delta t$. The state fields are decomposed as⁴

$$\begin{aligned} {}^{t+\Delta t}\mathbf{u} &= {}^t\mathbf{u} + \mathbf{u}; \quad {}^{t+\Delta t}{}_0\mathbf{S} = {}^t{}_0\mathbf{S} + {}_0\mathbf{S}; \quad {}_0\mathbf{S} = \boldsymbol{\phi}_{,\epsilon} {}_0\boldsymbol{\epsilon} \\ {}^{t+\Delta t}{}_0\boldsymbol{\epsilon} &= {}^t{}_0\boldsymbol{\epsilon} + {}_0\boldsymbol{\epsilon}, \quad {}^{t+\Delta t}{}_0\mathbf{f} = {}^t{}_0\mathbf{f} + {}_0\mathbf{f}; \quad {}^{t+\Delta t}{}_0\mathbf{T} = {}^t{}_0\mathbf{T} + {}_0\mathbf{T} \end{aligned} \quad (4)$$

where \mathbf{u} = increment in the displacement field
 ${}_0\mathbf{S}$ = increment in the Second Piola-Kirchhoff stress
 ${}_0\boldsymbol{\epsilon}$ = increment in the Green-Lagrange strain
 ${}_0\mathbf{f}$ = increment in the body force
 ${}_0\mathbf{T}$ = increment in the surface traction

Variation of the strain field is given as

$$\delta {}^{t+\Delta t}{}_0\boldsymbol{\epsilon} = \delta {}_0\boldsymbol{\epsilon} \quad (5)$$

The incremental strain field from Eq. (2) is given as

$${}_0\boldsymbol{\epsilon} = {}_0\mathbf{e} + {}_0\boldsymbol{\eta} \quad (6)$$

$${}_0\mathbf{e} = \frac{1}{2} [({}_0\mathbf{v}\mathbf{u}^T + ({}_0\mathbf{v}\mathbf{u}^T)^T + ({}_0\mathbf{v}\mathbf{u}^T) ({}_0\mathbf{v}^t\mathbf{u}^T)^T + ({}_0\mathbf{v}^t\mathbf{u}^T) ({}_0\mathbf{v}\mathbf{u}^T)^T)] \quad (7)$$

$${}_0\boldsymbol{\eta} = \frac{1}{2} [({}_0\mathbf{v}\mathbf{u}^T) ({}_0\mathbf{v}\mathbf{u}^T)^T] \quad (8)$$

Substituting Eqs. (4) - (6) in the virtual work principle, Eq. (1), written at $t+\Delta t$ and using the fact that state at t is in equilibrium, we obtain the following incremental virtual work principle:

$$\int ({}_0\mathbf{S} + {}_0\mathbf{S}) \cdot \delta {}_0\boldsymbol{\epsilon} {}^0dV - \int {}_0\mathbf{f} \cdot \delta \mathbf{u} {}^0dV - \int {}_0\mathbf{T} \cdot \delta \mathbf{u} {}^0d\Gamma_T = 0 \quad (9)$$

Equation (9) is still a nonlinear in incremental displacement field \mathbf{u} . It is linearized by assuming

$$\delta {}_0\boldsymbol{\epsilon} = \delta {}_0\mathbf{e}; \quad {}_0\mathbf{S} = \boldsymbol{\phi}_{,\epsilon} \cdot {}_0\mathbf{e} \quad (10)$$

and iteration is used within the load increment to satisfy the equilibrium exactly at $t+\Delta t$. The finite element procedure has been used to implement the preceding equations into a computer program ADINA.⁶

3. UNIFICATION OF DIMENSIONAL AND SHAPE DESIGN SENSITIVITY ANALYSIS

In the literature, shape and dimension design sensitivity analysis problems have been treated independently. In the shape problem, domain of the problem is allowed to vary whereas in the dimensional problem domain is fixed but cross-sectional dimensions are allowed to vary. It will be seen here that when variational formulation is used and volume integrals are used, there is no distinction between the two problems.

Consider the general functional requiring design sensitivity analysis:

$$\psi = \int_{V(b)} \bar{G}({}_0^t \mathbf{S}, {}_0^t \boldsymbol{\epsilon}, {}_0^t \mathbf{u}, b) dV + \int_{r_u(b)} \bar{g}({}_0^t \mathbf{T}, b) d\mathbf{r}_u + \int_{r_T(b)} \bar{h}({}_0^t \mathbf{u}, b) d\mathbf{r}_T \quad (11)$$

It can be seen that when design b is changed, the volume of the body as well as its surface change. As examples, consider optimal design of two simple bodies shown in Fig. 1. Are these shape or dimensional optimization problems? Our contention is that although length of the members is not treated as a design variable in these problems, volume of the body changes whenever any of the indicated design variables is changed. We must account for variations of the domain of the body while writing variations of the functional ψ in Eq. (11). Thus the variational concept for design sensitivity analysis is slightly different from the corresponding concept used in purely analysis problems where domain of the body remains fixed (at least in linear problems). This distinction is important in maintaining generality of the variational design sensitivity analysis theory where variation of the domain should be always considered.

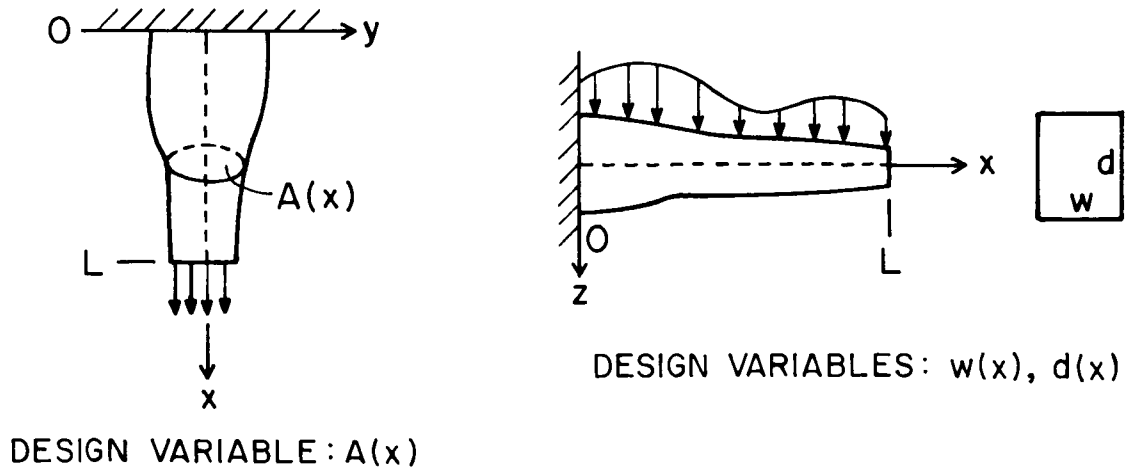


Figure 1. Examples of Optimal Design

4. CONCEPT OF REFERENCE VOLUME

The concept of a reference volume is extremely useful in problems where the volume of the body is changing. The idea, introduced recently in Ref. 7, is to map volume of the body in various configurations to a reference volume \bar{V} . This is shown in Fig. 2. The original volume of the body ${}^0V(b^0)$ moves to a volume ${}^tV(b^0)$ under a nonlinear motion. However, both the volumes can be mapped to the fixed reference volume \bar{V} under the mappings $F_1(b^0)$ and $F_2(b^0)$ respectively. The design process changes shape of the body so that its volume becomes ${}^0V(b^1)$ at the new design b^1 . This volume moves to ${}^tV(b^1)$ under the nonlinear motion. Both these volumes can also be mapped to the fixed reference volume \bar{V} .

The concept of reference volume is also quite useful in design sensitivity analysis. All the integrals of the problem are transformed to the reference volume using the proper transformation of the independent variables. The mapping to the fixed volume keeps changing under state or design variations. However, the reference volume never changes. Thus, when variations of various integrals are taken, the variations of the reference volume need not be considered. In numerical implementations, this concept is also very useful. It allows us to discretize the design problem into design elements that keep the same shape even when the real shape for

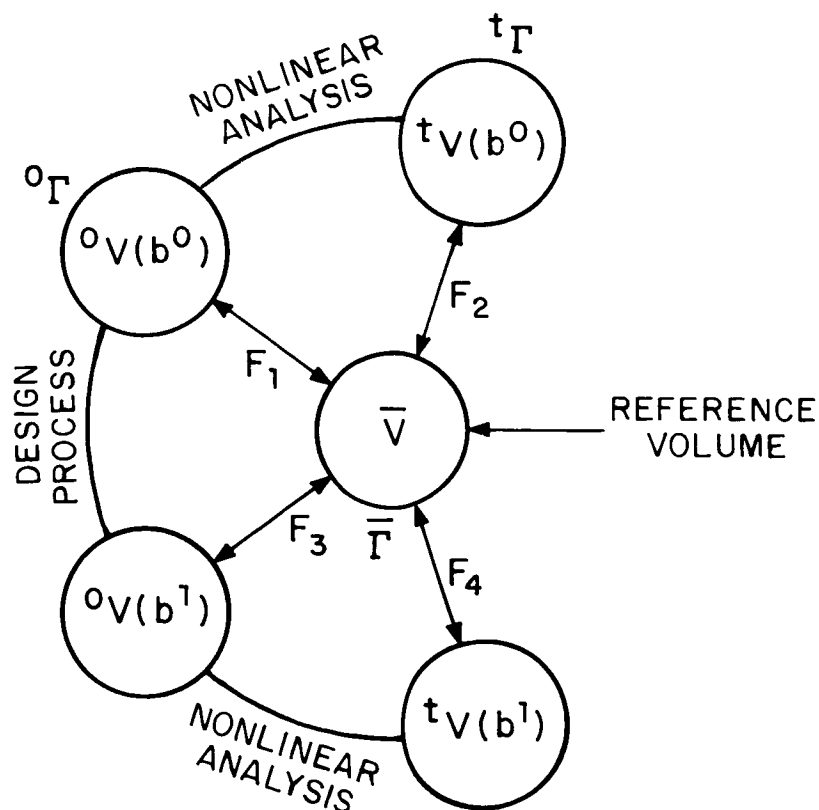


Figure 2. Concept of Reference Volume

the structure changes during the optimization process. Using the transformation of independent variables, various expressions are given as

Virtual Work Equation at Load Level t :

$$\int_0^t \mathbf{S}_0 \cdot \delta_0 \boldsymbol{\epsilon} J d\bar{V} - \int_0^t \mathbf{f}_0 \cdot \delta \mathbf{u} J d\bar{V} - \int_0^t \mathbf{T}_0 \cdot \delta \mathbf{u} \bar{J} d\bar{\Gamma}_T = 0 \quad (12)$$

Incremental Virtual Work Equation at Load Level $t+\Delta t$:

$$\int_0^t (\mathbf{S}_0 + \mathbf{S}) \cdot \delta_0 \boldsymbol{\epsilon} J d\bar{V} - \int_0^t \mathbf{f}_0 \cdot \delta \mathbf{u} J d\bar{V} - \int_0^t \mathbf{T}_0 \cdot \delta \mathbf{u} \bar{J} d\bar{\Gamma}_T = 0 \quad (13)$$

Green-Lagrange Strain Tensor:

$$\mathbf{e}_0^t = \frac{1}{2} [\bar{\mathbf{X}}^T (\mathbf{r} \nabla \mathbf{u}^T) + (\mathbf{r} \nabla \mathbf{u}^T)^T \bar{\mathbf{X}} + \bar{\mathbf{X}}^T (\mathbf{r} \nabla \mathbf{u}^T) (\mathbf{r} \nabla \mathbf{u}^T)^T \bar{\mathbf{X}}] \quad (14)$$

Incremental Strains:

$$\begin{aligned} \mathbf{e}_0 = \frac{1}{2} [\bar{\mathbf{X}}^T (\mathbf{r} \nabla \mathbf{u}^T) + (\mathbf{r} \nabla \mathbf{u}^T)^T \bar{\mathbf{X}} + \bar{\mathbf{X}}^T (\mathbf{r} \nabla \mathbf{u}^T) (\mathbf{r} \nabla \mathbf{u}^T)^T \bar{\mathbf{X}} \\ + \bar{\mathbf{X}}^T (\mathbf{r} \nabla \mathbf{u}^T) (\mathbf{r} \nabla \mathbf{u}^T)^T \bar{\mathbf{X}}] \end{aligned} \quad (15)$$

$$\mathbf{n}_0 = \frac{1}{2} [\bar{\mathbf{X}}^T (\mathbf{r} \nabla \mathbf{u}^T) (\mathbf{r} \nabla \mathbf{u}^T)^T \bar{\mathbf{X}}] \quad (16)$$

Functional for Sensitivity Analysis:

$$\psi = \int G(\mathbf{S}_0, \mathbf{e}_0, \mathbf{u}, b) J d\bar{V} + \int g(\mathbf{T}_0, b) \bar{J} d\bar{\Gamma}_u + \int h(\mathbf{u}, b) \bar{J} d\bar{\Gamma}_T \quad (17)$$

Jacobian of Transformation:

$$\mathbf{X} = \frac{\partial(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)}{\partial(\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_z)} ; \quad J = |\mathbf{X}| ; \quad \bar{\mathbf{X}} = \mathbf{X}^{-1} ; \quad \bar{J} = J |\bar{\mathbf{X}}^T \mathbf{n}| \quad (18)$$

In the above equations superscript or subscript r refers to the reference coordinates, \bar{J} is the area metric, and \mathbf{n} is the unit surface normal. Note that all quantities in the above integrals are functions of the reference coordinates. Also for oriented bodies such as bars and beams, J and $|\mathbf{X}|$ may be different from each other if we use volume integrals throughout the sensitivity analysis. This can be observed in the examples discussed later in the paper.

5. ADJOINT STRUCTURE APPROACH FOR GENERAL DESIGN SENSITIVITY ANALYSIS

Discrete form of the adjoint variable method has been discussed by several researchers.^{1,8-13} Variational form of the approach based on material derivative

concept is described in Ref. 13 where sensitivity with respect to shape variations is also considered. Adjoint structure approach is described in Refs. 14-17. The approach has been applied to some nonlinear and shape variation problems in Refs. 18-20. Recently, Belegundu²¹ has traced roots of the adjoint variable method to methods of sensitivity analysis in optimal control literature. In addition, he has shown that sensitivity analysis methods for static, dynamic, shape and distributed parameter problems can be viewed as the general Lagrange multiplier method. This shows that the adjoint variable is also a Lagrange multiplier for the state equations which gives a sensitivity interpretation for it.²² This interpretation is extremely useful and leads to some insights into the adjoint variable method. It also has implications in practical applications and numerical implementations of the method.

In the following derivation we combine the adjoint structure approach with the fixed reference volume concept to develop a general theory of design sensitivity analysis of linear or nonlinear structures. To avoid confusion, we use δ and $\bar{\delta}$ to indicate arbitrary variations of the state fields and variations with respect to design variable, respectively. Also, the notation $G_{,S}$ will be used to indicate partial derivative of G with respect to \mathbf{S}_0^t . Note that design sensitivity analysis is performed at the final state of the system denoted by left superscript t on various variables. Thus the virtual work equation (12) holds for the deformed configuration.

Now taking variation of the functional ψ in Eq. (17) with respect to design, we obtain

$$\begin{aligned} \bar{\delta}\psi = & \int \bar{\delta}G J d\bar{V} + \int G \bar{\delta}J d\bar{V} + \int \bar{\delta}g \bar{J} d\bar{\Gamma}_u \\ & + \int g \bar{\delta}\bar{J} d\bar{\Gamma}_u + \int h \bar{\delta}\bar{J} d\bar{\Gamma}_T + \int \bar{\delta}h \bar{J} d\bar{\Gamma}_T + \bar{\delta}\psi_I \end{aligned} \quad (19)$$

where $\bar{\delta}\psi_I$ represents implicit design variation of ψ given as

$$\begin{aligned} \bar{\delta}\psi_I = & \int (G_{,S} \cdot \bar{\delta}_0^t \mathbf{S} + G_{,\epsilon} \cdot \bar{\delta}_0^t \boldsymbol{\epsilon} + G_{,\mathbf{u}} \cdot \bar{\delta}_0^t \mathbf{u}) J d\bar{V} \\ & + \int g_{,T} \cdot \bar{\delta}_0^t \mathbf{T} \bar{J} d\bar{\Gamma}_u + \int h_{,\mathbf{u}} \cdot \bar{\delta}_0^t \mathbf{u} \bar{J} d\bar{\Gamma}_T \end{aligned} \quad (20)$$

The basic idea of the adjoint structure approach is to replace the implicit design variations of the state fields in Eq. (20) by explicit design variations and the adjoint state fields. To accomplish this we write design variations of various equations as follows:

Design Variations of the Constitutive Law (Eq. 3):

$$\bar{\delta}_0^t \mathbf{S} = \boldsymbol{\phi}_{,\epsilon} \cdot \bar{\delta}_0^t \boldsymbol{\epsilon} + \bar{\delta}\boldsymbol{\phi} \quad (21)$$

Design Variations of Strains:

$$\bar{\delta}_0^t \epsilon = \bar{\delta}_0^t \epsilon + \bar{\bar{\delta}}_0^t \epsilon \quad (22)$$

where

$$\begin{aligned} \bar{\delta}_0^t \epsilon = & \frac{1}{2} [\bar{\bar{X}}^T ({}_r \nabla \bar{\delta}^t \mathbf{u}^T) + ({}_r \nabla \bar{\delta}^t \mathbf{u}^T)^T \bar{\bar{X}} \\ & + \bar{\bar{X}}^T ({}_r \nabla \bar{\delta}^t \mathbf{u}^T) ({}_r \nabla \bar{\delta}^t \mathbf{u}^T)^T \bar{\bar{X}} + \bar{\bar{X}}^T ({}_r \nabla \bar{\delta}^t \mathbf{u}^T) ({}_r \nabla \bar{\delta}^t \mathbf{u}^T)^T \bar{\bar{X}}] \end{aligned} \quad (23)$$

$$\begin{aligned} \bar{\bar{\delta}}_0^t \epsilon = & \frac{1}{2} [\bar{\delta} \bar{\bar{X}}^T ({}_r \nabla \bar{\delta}^t \mathbf{u}^T) + ({}_r \nabla \bar{\delta}^t \mathbf{u}^T)^T \bar{\delta} \bar{\bar{X}} \\ & + \bar{\delta} \bar{\bar{X}}^T ({}_r \nabla \bar{\delta}^t \mathbf{u}^T) ({}_r \nabla \bar{\delta}^t \mathbf{u}^T)^T \bar{\bar{X}} + \bar{\delta} \bar{\bar{X}}^T ({}_r \nabla \bar{\delta}^t \mathbf{u}^T) ({}_r \nabla \bar{\delta}^t \mathbf{u}^T)^T \bar{\bar{X}}] \end{aligned} \quad (24)$$

Here $\bar{\delta}$ represents implicit design variations of the displacements and $\bar{\bar{\delta}}$ the explicit design variations of the strain fields.

Design Variations of Equilibrium Equation (12):

$$\begin{aligned} & \int \bar{\delta}_0^t \mathbf{S} \cdot \epsilon^a J d\bar{V} + \int {}^t \mathbf{S} \cdot \bar{\bar{\delta}} \epsilon^a J d\bar{V} + \int {}^t \mathbf{S} \cdot \epsilon^a \bar{\delta} J d\bar{V} - \int \bar{\delta}_0^t \mathbf{f} \cdot \mathbf{u}^a J d\bar{V} \\ & - \int {}^t \mathbf{f} \cdot \mathbf{u}^a \bar{\delta} J d\bar{V} - \int \bar{\delta}_0^t \mathbf{T} \cdot \mathbf{u}^a \bar{J} d\bar{\Gamma}_T - \int {}^t \mathbf{T} \cdot \mathbf{u}^a \bar{\delta} \bar{J} d\bar{\Gamma}_T = 0 \end{aligned} \quad (25)$$

where arbitrary variations of the primary state fields in Eq. (12) have been replaced by the corresponding fields for the adjoint structure denoted by the superscript 'a'. The adjoint structure and the corresponding state fields are defined later. Substitute for $\bar{\delta}_0^t \mathbf{S}$ from Eq. (21) into Eq. (25), use Eq. (22) and collect terms:

$$\begin{aligned} & \int [\epsilon^a \cdot \phi_{,\epsilon} (\bar{\delta}_0^t \epsilon + \bar{\bar{\delta}}_0^t \epsilon) + {}^t \mathbf{S} \cdot \bar{\bar{\delta}} \epsilon^a - \bar{\delta}_0^t \mathbf{f} \cdot \mathbf{u}^a + \epsilon^a \cdot \bar{\delta} \phi] J d\bar{V} \\ & - \int ({}^t \mathbf{f} \cdot \mathbf{u}^a - {}^t \mathbf{S} \cdot \epsilon^a) \bar{\delta} J d\bar{V} - \int (\bar{\delta}_0^t \mathbf{T} \cdot \mathbf{u}^a \bar{J} + {}^t \mathbf{T} \cdot \mathbf{u}^a \bar{\delta} \bar{J}) d\bar{\Gamma}_T = 0 \end{aligned} \quad (26)$$

Now, let us define the adjoint structure as follows:

Loads and Boundary Conditions:

$$\begin{aligned} \text{Initial strain field} & : \epsilon^{ai} = G_{,S} \\ \text{Initial stress field} & : \mathbf{S}^{ai} = G_{,\epsilon} \\ \text{Body force} & : \mathbf{f}^a = G_{,u} \\ \text{Specified Traction} & : \mathbf{T}^{a0} = h_{,u} \text{ on } \Gamma_T \\ \text{Specified Displacements} & : \mathbf{u}^{a0} = -g_{,T} \text{ on } \Gamma_u \end{aligned} \quad (27)$$

Constitutive Law (Linear):

$$\mathbf{S}^a = \boldsymbol{\phi}_{,\epsilon}^T \cdot (\boldsymbol{\epsilon}^a - \boldsymbol{\epsilon}^{ai}) - \mathbf{S}^{ai}; \quad \mathbf{S}^a = \text{the adjoint stress field.} \quad (28)$$

Virtual Work Equation:

$$\int \mathbf{S}^a \cdot \delta_0^t \boldsymbol{\epsilon} \, J \, d\bar{V} - \int \mathbf{f}^a \cdot \delta^t \mathbf{u} \, J \, d\bar{V} - \int \mathbf{T}^a \cdot \delta^t \mathbf{u} \, \bar{J} \, d\bar{\Gamma}_T = 0 \quad (29)$$

Substitute Eq. (28) into Eq. (29):

$$\begin{aligned} \int (\boldsymbol{\epsilon}^a \cdot \boldsymbol{\phi}_{,\epsilon} \cdot \delta_0^t \boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{ai} \cdot \boldsymbol{\phi}_{,\epsilon} \cdot \delta_0^t \boldsymbol{\epsilon} - \mathbf{S}^{ai} \cdot \delta_0^t \boldsymbol{\epsilon} - \mathbf{f}^a \cdot \delta^t \mathbf{u}) \, J \, d\bar{V} \\ - \int \mathbf{T}^a \cdot \delta^t \mathbf{u} \, \bar{J} \, d\bar{\Gamma}_T = 0 \end{aligned} \quad (30)$$

Strain Field (Linear in \mathbf{u}^a):

$$\begin{aligned} \boldsymbol{\epsilon}^a = \frac{1}{2} [\bar{\mathbf{X}}^T (\mathbf{r} \mathbf{v} \mathbf{u}^a)^T + (\mathbf{r} \mathbf{v} \mathbf{u}^a)^T \bar{\mathbf{X}} + \bar{\mathbf{X}}^T (\mathbf{r} \mathbf{v} \mathbf{u}^a)^T (\mathbf{r} \mathbf{v}^t \mathbf{u}^T)^T \bar{\mathbf{X}} \\ + \bar{\mathbf{X}}^T (\mathbf{r} \mathbf{v}^t \mathbf{u}^T) (\mathbf{r} \mathbf{v} \mathbf{u}^a)^T \bar{\mathbf{X}}] \end{aligned} \quad (31)$$

Substitute the adjoint equilibrium equation (30) into Eq. (26):

$$\begin{aligned} \int [\boldsymbol{\epsilon}^a \cdot \boldsymbol{\phi}_{,\epsilon} \cdot \bar{\delta}_0^t \boldsymbol{\epsilon} + {}_0^t \mathbf{S} \cdot \bar{\delta} \boldsymbol{\epsilon}^a - \bar{\delta}_0^t \mathbf{f} \cdot \mathbf{u}^a + \boldsymbol{\epsilon}^{ai} \cdot \boldsymbol{\phi}_{,\epsilon} \cdot \bar{\delta}_0^t \boldsymbol{\epsilon} + \boldsymbol{\epsilon}^a \cdot \bar{\delta} \boldsymbol{\phi} \\ + \mathbf{S}^{ai} \cdot \bar{\delta}_0^t \boldsymbol{\epsilon} + \mathbf{f}^a \cdot \bar{\delta}^t \mathbf{u}] \, J \, d\bar{V} - \int ({}_0^t \mathbf{f} \cdot \mathbf{u}^a - {}_0^t \mathbf{S} \cdot \boldsymbol{\epsilon}^a) \, \bar{\delta} J \, d\bar{V} \\ + \int (\mathbf{T}^a \cdot \bar{\delta}^t \mathbf{u} - \bar{\delta}_0^t \mathbf{T} \cdot \mathbf{u}^a) \, \bar{J} \, d\bar{\Gamma}_T - \int {}_0^t \mathbf{T} \cdot \mathbf{u}^a \, \bar{\delta} \bar{J} \, d\bar{\Gamma}_T = 0 \end{aligned} \quad (32)$$

Note that the variations of the state fields in Eq. (30) are arbitrary. So, they have been replaced as $\delta_0^t \boldsymbol{\epsilon} = \bar{\delta}_0^t \boldsymbol{\epsilon}$ and $\delta^t \mathbf{u} = \bar{\delta}^t \mathbf{u}$.

Substitute the adjoint loads from Eq. (27) into Eq. (20):

$$\begin{aligned} \bar{\delta} \psi_I = \int (\boldsymbol{\epsilon}^{ai} \cdot \bar{\delta}_0^t \mathbf{S} + \mathbf{S}^{ai} \cdot \bar{\delta}_0^t \boldsymbol{\epsilon} + \mathbf{f}^a \cdot \bar{\delta}^t \mathbf{u}) \, J \, d\bar{V} \\ + \int - \mathbf{u}^{a0} \cdot \bar{\delta}_0^t \mathbf{T} \, \bar{J} \, d\bar{\Gamma}_u + \int \mathbf{T}^{a0} \cdot \bar{\delta}^t \mathbf{u} \, \bar{J} \, d\bar{\Gamma}_T \end{aligned} \quad (33)$$

Substitute for $\bar{\delta}_0^t \mathbf{S}$ from Eq. (21) into Eq. (33):

$$\begin{aligned}\bar{\delta}\psi_I = & \int (\epsilon^{ai} \cdot \phi_{,\epsilon} \cdot \bar{\delta}_0^t \epsilon + \epsilon^{ai} \cdot \bar{\delta}\phi + S^{ai} \cdot \bar{\delta}_0^t \epsilon + f^a \cdot \bar{\delta}^t \mathbf{u}) J \, d\bar{V} \\ & + \int -u^{a0} \cdot \bar{\delta}_0^t T \bar{J} \, d\bar{\Gamma}_u + \int T^{a0} \cdot \bar{\delta}^t \mathbf{u} \bar{J} \, d\bar{\Gamma}_T\end{aligned}\quad (34)$$

Substitute for $\bar{\delta}_0^t \epsilon$ from Eq. (22) into Eq. (34):

$$\begin{aligned}\bar{\delta}\psi_I = & \int [(\epsilon^{ai} \cdot \phi_{,\epsilon} + S^{ai}) \cdot \bar{\delta}_0^t \epsilon + (\epsilon^{ai} \cdot \phi_{,\epsilon} + S^{ai}) \cdot \bar{\delta}_0^t \epsilon + \epsilon^{ai} \cdot \bar{\delta}\phi \\ & + f^a \cdot \bar{\delta}^t \mathbf{u}] J \, d\bar{V} + \int -u^{a0} \cdot \bar{\delta}_0^t T \bar{J} \, d\bar{\Gamma}_u + \int T^{a0} \cdot \bar{\delta}^t \mathbf{u} \bar{J} \, d\bar{\Gamma}_T\end{aligned}\quad (35)$$

Substitute Eq. (32) into Eq. (35) and use Eq. (28) to obtain

$$\begin{aligned}\bar{\delta}\psi_I = & \int [\bar{\delta}_0^t \mathbf{f} \cdot \mathbf{u}^a - (\epsilon^a - \epsilon^{ai}) \cdot \bar{\delta}\phi - S^a \cdot \bar{\delta}_0^t \epsilon - {}_0^t \mathbf{S} \cdot \bar{\delta} \epsilon^a] J \, d\bar{V} \\ & + \int ({}_0^t \mathbf{f} \cdot \mathbf{u}^a - {}_0^t \mathbf{S} \cdot \epsilon^a) \bar{\delta} J \, d\bar{V} + \int ({}_0^t \mathbf{T} \cdot \mathbf{u}^a \bar{\delta} \bar{J} + \bar{\delta}_0^t \mathbf{T} \cdot \mathbf{u}^a \bar{J}) d\bar{\Gamma}_T \\ & - \int \bar{\delta}_0^t \mathbf{T} \cdot \mathbf{u}^{a0} \bar{J} \, d\bar{\Gamma}_u\end{aligned}\quad (36)$$

Substitute Eq. (36) into Eq. (19):

$$\begin{aligned}\bar{\delta}\psi = & \int [\bar{\delta}_0^t \mathbf{f} \cdot \mathbf{u}^a - (\epsilon^a - \epsilon^{ai}) \cdot \bar{\delta}\phi - \mathbf{S}^a \cdot \bar{\delta}_0^t \epsilon - {}_0^t \mathbf{S} \cdot \bar{\delta} \epsilon^a + \bar{\delta} G] J \, d\bar{V} \\ & + \int ({}_0^t \mathbf{f} \cdot \mathbf{u}^a - {}_0^t \mathbf{S} \cdot \epsilon^a + G) \bar{\delta} J \, d\bar{V} + \int (\bar{\delta} g \bar{J} + g \bar{\delta} \bar{J} - \bar{\delta}_0^t \mathbf{T} \cdot \mathbf{u}^{a0} \bar{J}) d\bar{\Gamma}_u \\ & + \int [(h + {}_0^t \mathbf{T} \cdot \mathbf{u}^a) \bar{\delta} \bar{J} + (\bar{\delta} h + \bar{\delta}_0^t \mathbf{T} \cdot \mathbf{u}^a) \bar{J}] d\bar{\Gamma}_T\end{aligned}\quad (37)$$

Equation (37) is a general design sensitivity formula for linear and nonlinear structures (geometric and material nonlinearities), and shape, non-shape and material selection problems. Formula also gives sensitivity interpretations of the adjoint state fields. For example, it shows that the adjoint displacement field is sensitivity of functional ψ with respect to variations of the body force and surface tractions. This interpretation has been also derived in Refs. 21 and 22 for linear systems using the Lagrangian approach. Formula (37) also shows that the adjoint strain field gives variations of the functional ψ with respect to the constitutive law, the adjoint stress field is related to variations of ψ with respect to explicit design variations of the strain field, and variations of ψ with respect to variations of J can be recovered using adjoint and primary fields. These sensitivity interpretations will be observed in the example problems solved in the next section. These interpretations can be invaluable in practical applications and numerical implementations.

6. EXAMPLE PROBLEMS

Several analytical linear and nonlinear examples are solved to show use of Eq. (37) and interpretation of various terms. Although these examples are simple, they can be valuable in gaining insights into numerical implementation for larger complex problems. Also in using Eq. (37), we will use standard symbols σ for stress and ϵ for strain.

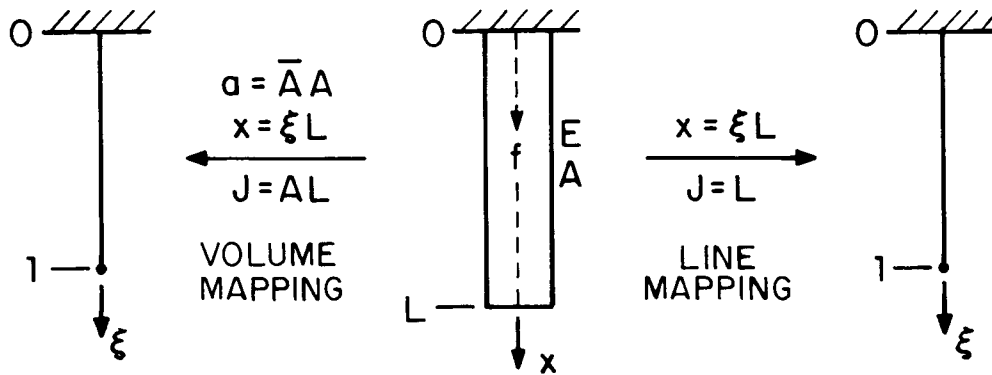
Example 1. Bar Under Self Weight

This example is taken from Ref. 7 where sensitivity of tip displacement with respect to length L is calculated. We will calculate sensitivities with respect to all parameters of the problem to demonstrate use of formula (37) for material, cross-sectional and length variations. The problem definition and various transformations are shown in Fig. 3. Small displacements and linear stress-strain law are assumed. The displacement field for the bar is given as $u(x) = fx(2L-x)/2E$ where f is the body force per unit volume. Thus

$$\begin{aligned} u(L) &= fL^2/E; \quad \bar{\delta}u(L) = (L^2/2E)\bar{\delta}f + (fL/E)\bar{\delta}L \\ &\quad + (0)\bar{\delta}A - (fL^2/2E^2)\bar{\delta}E \end{aligned} \quad (38)$$

There are at least two interpretations of this problem and both can be treated using Eq. (37).

First Interpretation. In this case, Eq. (37) can be interpreted as a line integral with x as the only independent variable. The stress-strain law of Eq. (3) must be interpreted as force-strain law, as the structure is only a line element. Note that this must be done with the formulas given in Refs. 14, 16, 18 and 20 when



DESIGN VARIABLES: f, E, A, L

Figure 3. Bar Under Self Weight

variations with respect to the cross-sectional area are needed. While using Eq. (37), the tip displacement can be treated as a boundary term or the interior term. We will use the latter approach. The functional for sensitivity analysis is given as

$$\psi = \int_0^1 u(\xi) J^{-1} \hat{\delta}(\xi-1) J d\xi; \quad G = u(\xi) J^{-1} \hat{\delta}(\xi-1); \quad G_{,u} = J^{-1} \hat{\delta}(\xi-1) \quad (39)$$

where $\hat{\delta}(\xi-1)$ is the Dirac delta function. The primary and adjoint fields can be obtained as

$$\begin{aligned} u(\xi) &= fL^2 \xi(2-\xi)/2E; & u^a(\xi) &= L\xi/EA \\ \epsilon(\xi) &= fL^2(1-\xi)/E; & \epsilon^a(\xi) &= L/EA \\ \epsilon &= \epsilon(\xi) J^{-1} = fL(1-\xi)/E; & \epsilon^a &= \epsilon^a(\xi) J^{-1} = 1/EA \\ N &= EA\epsilon = fAL(1-\xi); & N^a &= EA\epsilon^a = 1 \end{aligned} \quad (40)$$

where N is the axial force and $\phi = EA\epsilon$. Equation (37) reduces to

$$\begin{aligned} \bar{\delta}\psi &= \int_0^1 (\bar{\delta}\tilde{f}u^a - \epsilon^a \bar{\delta}\phi - N^a \bar{\delta}\epsilon - N \bar{\delta}\epsilon^a + \bar{\delta}G) J d\xi \\ &+ \int_0^1 (\tilde{f}u^a - N\epsilon^a + G) \bar{\delta}J d\xi \end{aligned} \quad (41)$$

Note that since we are using line integrals, the body force $\tilde{f} = fA$ must be used. Various quantities for use in Eq. (41) are

$$\begin{aligned} \bar{\delta}\phi &= (A\delta E + E\delta A) J^{-1} fL^2(1-\xi)/E; & \bar{\delta}\tilde{f} &= A\delta f + f\delta A \\ \bar{\delta}G &= -u(\xi) J^{-2} \hat{\delta}(\xi-1) \delta L; & \bar{\delta}\epsilon &= \epsilon(\xi) \bar{\delta}J^{-1} = -f(1-\xi) \delta L/E \\ \bar{\delta}\epsilon^a &= \epsilon^a(\xi) \bar{\delta}J^{-1} = -\delta L/EAL \end{aligned} \quad (42)$$

Substituting all the quantities in Eq. (41) and carrying out the integrations, we obtain the required sensitivity equation which is the same as Eq. (38). The sensitivity interpretations of the adjoint fields can be directly observed.

Second Interpretation. In this case, Eq. (3) will be treated as a volume integral. The functional for sensitivity analysis is given as

$$\psi = \int_0^1 \int_{\bar{A}} (AL)^{-1} u(\xi) \hat{\delta}(\xi-1) J d\bar{A} d\xi; \quad G = (AL)^{-1} u(\xi) \hat{\delta}(\xi-1) \quad (43)$$

The displacement and strain fields are the same as given in Eq. (40). However, the stress-strain law is the usual Hooke's Law:

$$\sigma = E\epsilon = fL(1-\xi); \quad \sigma^a = 1/A \quad (44)$$

Equation (37) reduces to

$$\begin{aligned} \bar{\delta}\psi = & \int_0^1 \int_{\bar{A}} (\bar{\delta}fu^a - \epsilon^a \bar{\delta}\phi - \sigma^a \bar{\delta}\epsilon - \sigma \bar{\delta}\epsilon^a + \bar{\delta}G) J d\bar{A} d\xi \\ & + \int_0^1 \int_{\bar{A}} (fu^a - \sigma \epsilon^a + G) \bar{\delta}J d\bar{A} d\xi \end{aligned} \quad (45)$$

Various quantities for use in Eq. (45) are

$$\bar{\delta}\phi = fL\delta E(1-\xi)/E; \quad \bar{\delta}J = L\delta A + A\delta L; \quad \bar{\delta}G = -(A^{-1} + L^{-1})G \quad (46)$$

Substituting various quantities from Eqs. (40), (42) and (46) into Eq. (45), we again obtain the sensitivity expression given in Eq. (38). The sensitivity interpretation of the adjoint fields can be easily observed.

Example 2. Cantilever Beam

This example is also discussed in Ref. 7 where sensitivity of tip deflection with respect to the length is given. Figure 4 defines the problem and the transformations to the reference volume. The design variables are chosen as $b = (E, s, h, L)$.

The tip deflection using small displacements beam theory is given as $w(L) = PL^3/3EI$ and its variation with respect to the design variables is given as

$$\bar{\delta}w(L) = -\frac{PL^3}{3E^2I} \bar{\delta}E - \frac{PL^3h^3}{36EI^2} \bar{\delta}s - \frac{PL^3sh^2}{12EI^2} \bar{\delta}h + \frac{PL^2}{EI} \bar{\delta}L \quad (47)$$

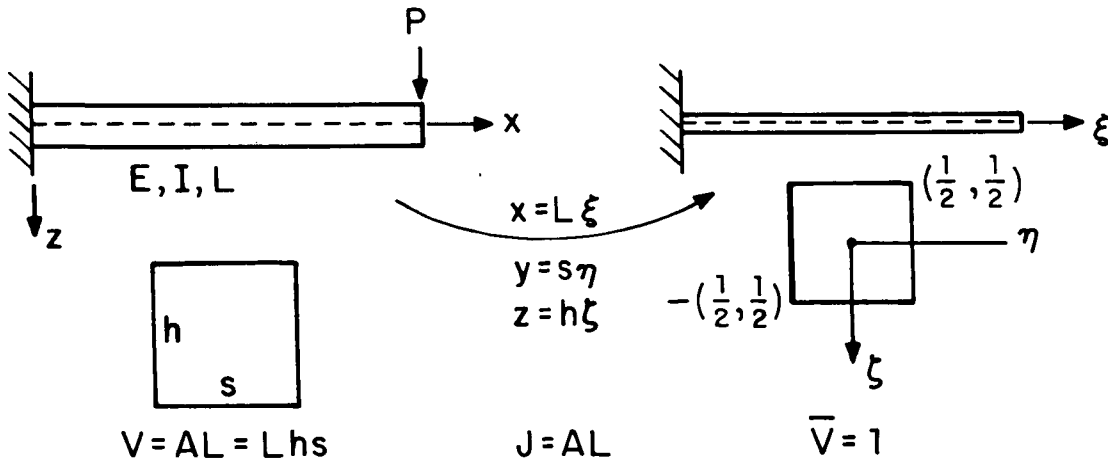


Figure 4. Cantilever Beam

The functional for the tip deflection and the function G are given as

$$\psi = \int_0^1 \int_{\bar{A}} (AL)^{-1} w(\xi) \hat{\delta}(\xi-1) AL d\bar{A} d\xi \quad (48)$$

$$G = (AL)^{-1} w(\xi) \hat{\delta}(\xi-1); \quad G_{,w} = (AL)^{-1} \hat{\delta}(\xi-1) \quad (49)$$

The primary and adjoint structure solutions are given

$$w_{,\xi\xi} = \frac{PL^3}{EI} (1-\xi); \quad w(\xi) = \frac{PL^3 \xi^2}{6EI} (3-\xi) \quad (50)$$

$$w^a_{,\xi\xi} = \frac{L^3}{EI} (1-\xi); \quad w^a(\xi) = \frac{L^3 \xi^2}{6EI} (3-\xi) \quad (51)$$

The sensitivity formula of Eq. (37) is reduced to

$$\begin{aligned} \bar{\delta}\psi = & \int_0^1 \int_{\bar{A}} (-\epsilon^a \bar{\delta}\phi - \sigma \bar{\delta}\epsilon^a - \sigma^a \bar{\delta}\epsilon + \bar{\delta}G) AL d\bar{A} d\xi \\ & + \int_0^1 \int_{\bar{A}} (-\sigma \epsilon^a + G) \bar{\delta}(AL) d\bar{A} d\xi \end{aligned} \quad (52)$$

The following quantities are needed to complete integrations in Eq. (52):

$$\epsilon = \zeta w_{,\xi\xi} hL^{-2}; \quad \epsilon^a = \zeta w^a_{,\xi\xi} hL^{-2}; \quad \bar{\delta}G = \hat{w} \hat{\delta}(\xi-1) \bar{\delta}(AL)^{-1}$$

$$\sigma = E\epsilon; \quad \bar{\delta}\phi = \epsilon \delta E; \quad \sigma^a = E \epsilon^a$$

$$\bar{\delta}\epsilon = \zeta w_{,\xi\xi} \bar{\delta}(hL^{-2}); \quad \bar{\delta}\epsilon^a = \zeta w^a_{,\xi\xi} \bar{\delta}(hL^{-2})$$

Substituting these quantities in Eq. (52) and carrying out the integrations we get the sensitivity expression given in Eq. (47). It is interesting to again note that the adjoint displacement field given in Eq. (51) represents the sensitivity of the primary displacement field (Eq. 52) with respect to the load parameters P; i.e. $u^a(\xi) = d\psi/dP$.

Example 3. Materially Nonlinear Problem

Consider the bar of Fig. 3 subjected to a load P in the x direction at the free end. The material for the bar obeys a nonlinear stress-strain law $\sigma = E\epsilon^{1/2} (\epsilon > 0)$; so $\phi = E\epsilon^{1/2}$. We will consider E, A and L as design variables and determine sensitivity of the tip deflection. Transformation to the reference volume gives $x = L\xi$, $a = AA$, $J = AL$, $V = AL$, and $\bar{V} = 1$. Nonlinear analysis of the primary structure yields:

$$u(\xi) = \frac{P^2 L \xi}{A^2 E^2}; \quad \bar{u}(L) = -\frac{2P^2 L}{A^2 E^3} \bar{\delta} E - \frac{2P^2 L}{A^3 E^2} \bar{\delta} A + \frac{P^2}{A^2 E^2} \bar{\delta} L \quad (53)$$

The functional for sensitivity analysis is given as

$$\psi = \int_0^1 \int_{\bar{A}} \bar{A}^{-1} L^{-1} u(\xi) \hat{\delta}(\xi-1) (AL) d\bar{A} d\xi \quad (54)$$

$$G = A^{-1} L^{-1} u(\xi) \hat{\delta}(\xi-1); \quad G_{,u} = A^{-1} L^{-1} \hat{\delta}(\xi-1) \quad (55)$$

The adjoint structure is linear with the stress-strain law as

$$\sigma^a = \phi_{,\epsilon} \epsilon^a = \frac{1}{2} E \epsilon^{1/2} \epsilon^a = \frac{AE^2}{2P} \epsilon^a \quad (56)$$

The equilibrium equation for the adjoint structure gives

$$u^a(\xi) = \frac{2PL\xi}{A^2 E^2}, \quad u_{,\xi}^a = \frac{2PL}{A^2 E^2} \quad (57)$$

The sensitivity formula of Eq. (37) reduces to

$$\begin{aligned} \bar{\delta}\psi = & \int_0^1 \int_{\bar{A}} (-\epsilon^a \bar{\delta}\phi - \sigma^a \bar{\delta}\epsilon - \sigma \bar{\delta}\epsilon^a + \bar{\delta}G) J d\bar{A} d\xi \\ & + \int_0^1 \int_{\bar{A}} (-\sigma \epsilon^a + G) \bar{\delta} J d\bar{A} d\xi \end{aligned} \quad (58)$$

Various quantities for Eq. (58) are

$$\epsilon = u_{,\xi} L^{-1} = \frac{P^2 L}{A^2 E^2}; \quad \epsilon^a = u_{,\xi}^a L^{-1} = \frac{2PL}{A^2 E^2}; \quad \bar{\delta}G = u \hat{\delta}(\xi-1) \bar{\delta}(AL)^{-1}$$

$$\sigma = E \epsilon^{1/2}; \quad \phi_{,\epsilon} = (1/2) E \epsilon^{-1/2}, \quad \bar{\delta}\phi = \epsilon^{1/2} \bar{\delta}E; \quad \sigma^a = \phi_{,\epsilon} \epsilon^a$$

$$\bar{\delta}\epsilon = u_{,\xi} \bar{\delta}L^{-1} = -L^{-2} u_{,\xi} \delta L; \quad \bar{\delta}\epsilon^a = -L^{-2} u_{,\xi}^a \delta L$$

Substituting these quantities into Eq. (58), we obtain the sensitivity formula of Eq. (53). It is interesting to observe sensitivity meaning of the adjoint displacement field in Eq. (57); i.e. $u^a(\xi) = d\psi/dP$.

Example 4. Geometrically Nonlinear Problem

Consider the two bar structure shown in Fig. 5. The material for the structure is linear, so $\sigma = E\epsilon$. Transformation to the reference volume is shown in the

figure. The design variables for the problem are $\mathbf{b} = (E, A, L)$. The strain for the problem is given as

$$t_o \epsilon = (t_L - o_L)/o_L = \frac{1}{2} w^2 L^{-2} \equiv \epsilon \quad (59)$$

The deflection at the center and member strains are calculated as

$$w = \frac{P^{1/3} L}{(EA)^{1/3}}; \quad \bar{\epsilon} = \epsilon L^2 = \frac{1}{2} w^2 = \frac{P^{2/3} L^2}{(EA)^{2/3}} \quad (60)$$

The incremental equilibrium equation in terms of displacement at the center is $3EAw^2 L^{-3} \delta w = \delta P$. The functional for sensitivity analysis is given as

$$\psi = \int_0^1 \int_{\bar{A}} (AL)^{-1} w(\xi) \hat{\delta}(\xi-1) AL d\bar{A} d\xi \quad (61)$$

$$G = (AL)^{-1} w(\xi) \hat{\delta}(\xi-1); \quad G_{,w} = (AL)^{-1} \hat{\delta}(\xi-1) \quad (62)$$

The equilibrium equation for the adjoint structure (using the incremental equilibrium equation of the primary structure) is given as

$$3EAL^{-3} w^2 w^a = \int_0^1 \int_{\bar{A}} (AL)^{-1} \hat{\delta}(\xi-1) AL d\bar{A} d\xi = 1$$

$$w^a = \frac{L^3}{3EAw^2} = \frac{L}{3E^{1/3} A^{1/3} P^{2/3}} \quad (63)$$

Total axial displacement and displacement at any point are given as

$$u^a(L) = w^a \sin \theta = w^a w L^{-1}; \quad u^a(\xi) = w^a w L^{-1} \xi \quad (64)$$

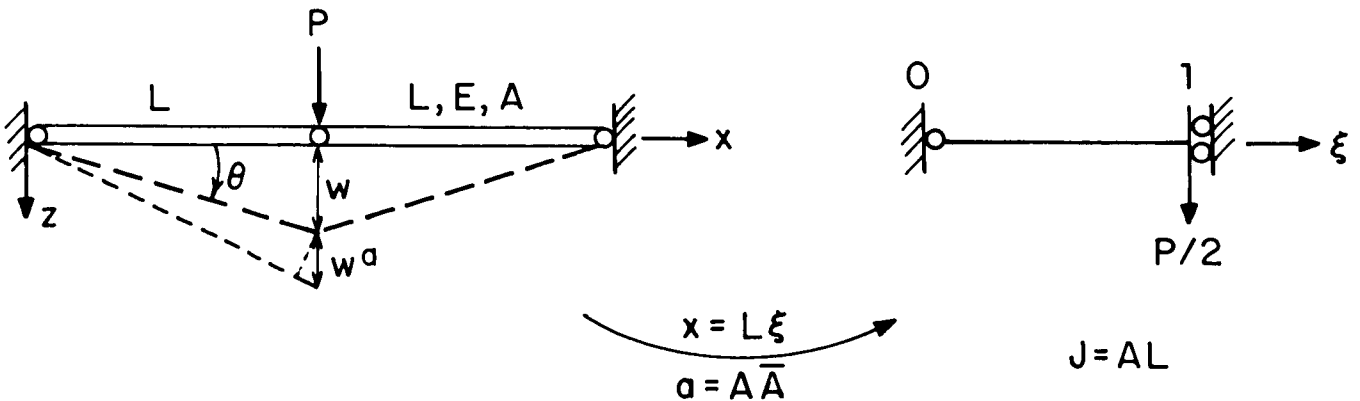


Figure 5. Two Member Structure

The adjoint strains are given as

$$\epsilon^a = w^a w_L^{-2} = \bar{\epsilon}^a L^{-2}, \quad \bar{\epsilon}^a = w^a w \quad (65)$$

The sensitivity formula of Eq. (37) reduces to

$$\begin{aligned} \bar{\delta}\psi = & 2 \int_0^1 \int_{\bar{A}} (-\epsilon^a \bar{\delta}\phi - \sigma^a \bar{\delta}\epsilon - \sigma \bar{\delta}\epsilon^a + \bar{\delta}G) J d\bar{A} d\xi \\ & + 2 \int_0^1 \int_{\bar{A}} (-\sigma \epsilon^a + G) \bar{\delta}J d\bar{A} d\xi \end{aligned} \quad (66)$$

Note that factor of 2 is used because volume integrals in Eq. (37) are for the entire structure. Various quantities for Eq. (66) are

$$\begin{aligned} \bar{\delta}\phi = & \epsilon \delta E; \quad \bar{\delta}\epsilon = \bar{\epsilon} \delta L^{-2} = -w^2 L^{-3} \delta L; \quad \bar{\delta}\epsilon^a = \bar{\epsilon}^a \delta L^{-2} = -2w w^a L^{-3} \delta L \\ \sigma = & E\epsilon = \frac{1}{2} E w^2 L^{-2}; \quad \sigma^a = E\epsilon^a = E w w^a L^{-2}; \quad \bar{\delta}J = L\delta A + A\delta L \end{aligned} \quad (67)$$

Substituting these quantities into Eq. (66), we get

$$\bar{\delta}\psi = - \frac{P^{1/3} L}{3E^{4/3} A^{1/3}} \bar{\delta}E - \frac{P^{1/3} L}{3E^{1/3} A^{4/3}} \bar{\delta}A + (P/EA)^{1/3} \bar{\delta}L \quad (68)$$

which can also be obtained directly from Eq. (60). Comparing w and w^a in Eqs. (60) and (63), we again observe the sensitivity interpretation of the adjoint displacement field.

Next, consider the member stress given in Eq. (67) as $\sigma = (P^{2/3} E^{1/3}) / (2A^{2/3})$. Its design sensitivity is given as

$$\bar{\delta}\sigma = \frac{P^{2/3}}{6E^{2/3} A^{2/3}} \bar{\delta}E - \frac{P^{2/3} E^{1/3}}{3A^{5/3}} \bar{\delta}A + (0) \bar{\delta}L \quad (69)$$

The functional for design sensitivity analysis is given as

$$\psi = \int_0^1 \int_{\bar{A}} (AL)^{-1} \sigma AL d\bar{A} d\xi = \int_0^1 \int_{\bar{A}} G(\sigma) AL d\bar{A} d\xi \quad (70)$$

The adjoint load G, \mathbf{u} in this case is zero but initial strain in the adjoint structure and stress strain law are given as $\epsilon^{ai} = G, \sigma = (AL)^{-1}$, $\sigma^a = E(\epsilon^a - \epsilon^{ai})$. The adjoint equilibrium equation in terms of central displacement gives $w^a = (wA)^{-1} L/3$ and $\epsilon^a = w w^a L^{-2} = (AL)^{-1}/3$. Substituting appropriate quantities in Eq. (37), it can be verified that Eq. (69) is obtained. It can be also directly verified that $w^a = d\psi/dP \equiv d\sigma/dP$.

7. CONCLUDING REMARKS

A general formula for design sensitivity analysis of linear and nonlinear structures using variational approach has been developed. Equations of continuum mechanics are used and the concepts of reference volume and adjoint structure are exploited. Use of the formula is demonstrated on a few simple analytical problems. The theory can be easily adapted for finite element modelling of structures. The finite element models for the primary and adjoint structures can be independent of each other. For modelling of design optimization problems, the concept of a reference volume is translated to the concept of a design element that is invariant with respect to design changes. These observations can have considerable implications in numerical implementations for design sensitivity analysis and optimization of complex structures.

Considerable numerical work has been done for design sensitivity analysis and optimization of linear structures.¹ Material derivative approach has been exploited for shape optimization. In this regard recent work of Choi and Co-workers,²³ Yang and Co-workers²⁴ and Hou and Co-workers²⁵ is significant. Yang and Co-workers²⁴ have shown equivalence of variational and finite element formulations of design sensitivity analysis of shape problems for linear structures. This equivalence can also be shown for nonlinear problems. Hou and Co-workers²⁵ have discussed some difficulties with the material derivative approach of design sensitivity analysis of linear shape problems. They have suggested numerical procedures to improve accuracy of the approach.

Design sensitivity analysis and optimization with nonlinear response is just beginning to be studied. Finite element approach for nonlinear stresses, strains, displacements and the buckling load has been recently studied.²⁶⁻³² More research needs to be done to fully develop this area.

REFERENCES

- ¹Adelman, H.M. and Haftka, R.T., "Sensitivity Analysis of Discrete Structural Systems," AIAA Journal, Vol. 24, No. 5, May 1986, pp. 823-832.
- ²Adelman, H.M. and Haftka, R.T. (Eds.), "Sensitivity Analysis in Engineering," Proceedings of the NASA Symposium, NASA Langley Research Center, Hampton, VA, NASA CP 2457, 1987.
- ³Gadala, M.S., Dokainish, M.A., and Oravas, A.E.G., "Formulation Methods of Geometric and Material Nonlinear Problems," International Journal for Numerical Methods in Engineering, Vol. 20, 1984, pp. 887-914.
- ⁴Bathe, K.J., Finite Element Procedures in Engineering Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1982.

⁵Ryu, Y.S. and Arora, J.S., "Review of Nonlinear FE Methods with Substructures," Journal of Engineering Mechanics Division, ASCE, Vol. 111, No. 11, November 1985, pp. 1361-1379.

⁶Bathe, K.J. (Ed.), "Proceedings of the 5th ADINA Conference," Computers and Structures, Vol. 21, Nos. 1/2, 1985.

⁷Haber, R.B., "Application of The Eulerian Lagrangian Kinematic Description to Structural Shape Optimization," Proceedings of NATO Advanced Study Institute on Computer-Aided Optimal Design, C.A. Mota Soares (Ed.), Troia, Portugal, June 29 - July 11, 1986, pp. 297-307.

⁸Arora, J.S. and Govil, A.K., "Design Sensitivity Analysis with Substructuring," Journal of Engineering Mechanics Division, ASCE, Vol. 103, No. EM4, 1977, pp. 537-548.

⁹Arora, J.S. and Haug, E.J., "Methods of Design Sensitivity Analysis in Structural Optimization," AIAA Journal, Vol. 17, Sept. 1979, pp. 970-974.

¹⁰Haug, E.J. and Arora, J.S., "Design Sensitivity Analysis of Elastic Mechanical Systems," Computer Methods in Applied Mechanics and Engineering, Vol. 15, 1978, pp. 35-62.

¹¹Haug, E.J. and Arora, J.S., Applied Optimal Design, Wiley-Interscience, John Wiley and Sons, New York, N.Y. 1979.

¹²Hsieh, C.C. and Arora, J.S., "Structural Design Sensitivity Analysis with General Boundary Conditions: Static Problem," International Journal for Numerical Methods in Engineering, Vol. 20, 1984, pp. 1661-1670.

¹³Haug, E.J., Choi, K.K., and Komkov, V., Design Sensitivity Analysis of Structural Systems, Academic Press, Orlando, FL, 1986.

¹⁴Dems, K. and Mroz, Z., "Variational Approach by Means of Adjoint Systems to Structural Optimization and Sensitivity Analysis - I. Variation of Material Parameters Within Fixed Domain," International Journal of Solids and Structures, Vol. 19, No. 8, 1983, pp. 677-692.

¹⁵Dems, K. and Mroz, Z., "Variational Approach by Means of Adjoint Systems to Structural Optimization and Sensitivity Analysis - II: Structure Shape Variation," International Journal of Solids and Structures, Vol. 6, 1984, pp. 527-552.

¹⁶Dems, K. and Mroz, Z., "Variational Approach to First- and Second-Order Sensitivity Analysis of Elastic Structures," International Journal for Numerical Methods in Engineering, Vol. 21, 1985, pp. 637-661.

¹⁷Dems, K., "Sensitivity Analysis in Thermoelasticity," Proceedings of NATO Advanced Study Institute on Computer-Aided Optimal Design, C.A. Mota Soares (Ed.), Troia, Portugal, June 29 - July 11, 1986, pp. 287-297.

¹⁸Haftka, R.T. and Mroz, Z., "First and Second Order Sensitivity Analysis of Linear and Nonlinear Structures," AIAA Journal, Vol. 24, No. 7, July 1986, pp. 1187-1192.

¹⁹Mroz, Z., Kamat, M.P., and Plant, R.H., "Sensitivity Analysis and Optimal Design of Nonlinear Beams and Plates," Journal of Structural Mechanics, Vol. 13, Nos. 3/4, 1985, pp. 245-266.

²⁰Mroz, Z., "Sensitivity Analysis and Optimal Design with Account for Varying Shape and Support Conditions," Proceedings of NATO Advanced Study Institute on Computer-Aided Optimal Design, C.A. Mota Soares (Ed.), Troia, Portugal, June 29 - July 11, 1986, pp. 109-144.

²¹Belegundu, A.D., "Lagrangian Approach to Design Sensitivity Analysis," Journal of Engineering Mechanics Division, ASCE, Vol. 111, No. 5, May 1985, pp. 680-695.

²²Belegundu, A.D. and Arora, J.S., "A Sensitivity Interpretation of Adjoint Variables in Optimal Design," Computer Methods in Applied Mechanics and Engineering, Vol. 48, 1985, pp. 81-89.

²³Choi, K.K., "Shape Design Sensitivity Analysis and Optimal Design of Structural Systems," Proceedings of NATO Advanced Study Institute on Computer-Aided Optimal Design, C.A. Mota Soares (Ed.), Troia, Portugal, June 29 - July 11, 1986, pp. 54-108.

²⁴Yang, R.J. and Botkin, M.E., "Comparison between the Variational and Implicit Differentiation Approaches to Shape Design Sensitivities," AIAA Journal, Vol. 24, No. 6, June 1986, pp. 1027-1032.

²⁵Hou, J.W., Chen, J.L., and Sheen, J.S., "Computational Method for Optimization of Structural Shapes," AIAA Journal, Vol. 24, No. 6, June 1986, pp. 1005-1012.

²⁶Ryu, Y.S., Haririan, M., Wu, C.C., and Arora, J.S., "Methods of Design Sensitivity Analysis for Nonlinear Response," Computers and Structures, Vol. 21, No. 1/2, 1985, pp. 245-255.

²⁷Haririan, M. and Arora, J.S., "Use of ADINA for Design Optimization of Nonlinear Structures," Computers and Structures, 1987.

²⁸Wu, C.C. and Arora, J.S., "Simultaneous Analysis and Design Optimization of Nonlinear Response," Engineering with Computers, 1987.

²⁹Wu, C.C. and Arora, J.S., "Design Sensitivity Analysis of Nonlinear Response Using Incremental Procedure," AIAA Journal, 1987.

³⁰Wu, C.C. and Arora, J.S., "Design Sensitivity Analysis of Nonlinear Buckling Load," Computational Mechanics, 1987.

³¹Arora, J. and Wu, C.C., "Design Sensitivity Analysis of Nonlinear Structures," Proceedings of NATO Advanced Study Institute on Computer-Aided Optimal Design, C.A. Mota Soares (Ed.), Troia, Portugal, June 29 - July 11, 1986, pp. 228-246.

³²Kamat, M.P., "Optimization of Shallow Arches Against Instability Using Sensitivity Derivatives," Proceedings of the NASA Symposium on Sensitivity Analysis in Engineering, H.M. Adelman and R.T. Haftka (Eds.), Langley Research Center, Hampton, VA, NASA CP 2457, 1987.